Phys 410
Spring 2013
Lecture #33 Summary
15 April, 2013

In the last lecture we arrived at the most general description of the rotational motion of a rigid body by relating the angular momentum to the angular velocity of the object as: $\vec{L} = \bar{\bar{I}}\vec{\omega}$, where $\vec{L} = \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix}$ is the angular momentum represented as a column vector, $\bar{\bar{I}} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$ is called the inertia tensor, and $\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$ is the angular velocity vector. Note that the inertia tensor is symmetric about the diagonal: $I_{ij} = I_{ji}$. .

We considered the motion of a "top" that was set into motion at angular velocity ω along one of its principal axes and then supported at a single point on its rotation axis. The top is supported on one of its principal axes, which we will call the 3-axis, with direction $\widehat{e_3}$. The top is observed to precess in a cone around the vertical direction $\hat{z}$. We can write the angular momentum as $\vec{L} = \lambda_3 \omega \widehat{e_3}$, where λ_3 is the principal moment for this axis. There are two forces acting on the top, the normal force at the point of suspension, and the weight, acting on the center of mass. We take the origin to be at the point of suspension so that only the weight exerts a torque. The torque leads to a time rate of change of the angular momentum: $\vec{\Gamma} = \dot{\vec{L}}$. The torque is $\vec{\Gamma} = \vec{R} \times M\vec{g}$, which points in a direction perpendicular to $\widehat{e_3}$, and therefore $\vec{L}$. This means that $|\vec{L}|$ remains fixed, but the direction of $\vec{L}$ will change. We found that $\dot{\widehat{e_3}} = \vec{\Omega} \times \widehat{e_3}$, where $\vec{\Omega} = \frac{RMg}{\lambda_3 \omega}\hat{z}$, showing that the principal axis of the top $\widehat{e_3}$ is rotating around the $\hat{z}$ axis at angular velocity $\frac{RMg}{\lambda_3 \omega}$.

We then considered the description of Newton's second law from the perspective of an observer on the rotating object. The observer in the "body frame" can identify the principal axes of the object and describe the angular momentum using the diagonalized inertia tensor as $\vec{L} = (\lambda_1 \omega_1, \lambda_2 \omega_2, \lambda_3 \omega_3)$. An inertial observer in the "space frame" is in position to identify correctly the net torque $\vec{\Gamma}$ acting on the angular momentum vector, and to write Newton's second law of motion (in rotational form) as $\vec{\Gamma} = \left(\frac{d\vec{L}}{dt}\right)_{space}$. We learned how to translate the time derivative of a vector quantity from an inertial frame to a rotating reference frame in Lecture 14: $\left(\frac{d\vec{Q}}{dt}\right)_{Space} = \left(\frac{d\vec{Q}}{dt}\right)_{Body} + \vec{\Omega} \times \vec{Q}$, where $\vec{Q}$ is the vector in question and the non-inertial reference frame is rotating with angular velocity $\vec{\Omega}$. In this case we can write the equations of motion as witnessed in the body frame as $\vec{\Gamma} = \left(\frac{d\vec{L}}{dt}\right)_{Body} + \vec{\omega} \times \vec{L}$, which translates in component form into the Euler equations:

$$\Gamma_1 = \lambda_1 \dot{\omega}_1 - \omega_2 \omega_3 (\lambda_2 - \lambda_3)$$

$$\Gamma_2 = \lambda_2 \dot{\omega}_2 - \omega_1 \omega_3 (\lambda_3 - \lambda_1)$$

$$\Gamma_3 = \lambda_3 \dot{\omega}_3 - \omega_1 \omega_2 (\lambda_1 - \lambda_2)$$

This set of equations describes how the angular velocity vector evolves as it is acted upon by a net external torque. When applied to the case of the spinning top discussed above, we note that $\Gamma_3 = 0$ (the torque acts in a direction perpendicular to $\widehat{e_3}$) and $\lambda_1 = \lambda_2$, hence $\lambda_3 \dot{\omega}_3 = 0$, so that ω_3 is constant. Thus the angular velocity vector remains aligned with 3-axis and no other component of $\vec{\omega}$ is excited.